A Logic of Visibility, Perception, and Knowledge: Completeness and Correspondence Results

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Abstract

$\mathcal{VSK}$ logic is a family of multi-modal logics for reasoning about the information properties of computational agents situated in some environment. Using $\mathcal{VSK}$ logic, we can represent what is *objectively true* of the environment, the information that is *visible*, or *knowable* about the environment, information the agent *perceives* of the environment, and finally, information the agent actually *knows* about the environment. The semantics of $\mathcal{VSK}$ logic are given in terms of a general, automata-like model of agents. In this paper, we prove completeness for an axiomatisation of $\mathcal{VSK}$ logic, and present correspondence results for a number of $\mathcal{VSK}$ interaction axioms in terms of the architectural properties of the agent that they represent. This completeness proof is novel in that we are able to prove completeness with respect to the automata-like semantics of the formalism. We give an example to illustrate the formalism, and present conclusions and issues for further work.

Topic Area: Logics for Practical Reasoning

1 Introduction

When designing an agent to carry out a task in some environment, it is frequently necessary to reason about the *information properties* of the agent and its environment. For example, many tasks depend on an agent being able to *access* certain information in the environment. If this information is not accessible, then we will not be able to implement an agent to carry out the desired task. Similarly, knowing that a particular piece of information is essential for some task gives us a functional requirement for any agent that will carry out the task: the agent's sensors must be capable of *perceiving* this information. Finally, many applications demand the ability to store and reason about information from the environment.

In this paper, we present a logic that allows us to capture such information properties. $\mathcal{VSK}$ *logic* allows us to represent what is *objectively true* of an environment, what is *visible*, or *knowable* about the environment, what an agent *perceives* of the environment, and finally, what the agent actually *knows* about the environment. Syntactically, $\mathcal{VSK}$ logic is a propositional multi-modal logic, containing modalities "$\mathcal{V}$", "$\mathcal{S}$", and "$\mathcal{K}$", where $\mathcal{V}\varphi$ means that the information φ is accessible in the current environment state; $\mathcal{S}\varphi$ means that the agent perceives φ; and $\mathcal{K}\varphi$ means that the agent knows φ.

A key feature of $\mathcal{VSK}$ logic is that its semantics are given with respect to a simple, general model of agents and their environments. We are able to characterise possible axioms of $\mathcal{VSK}$ logic with respect to this semantic model, although we do it by using standard canonical model constructions. Consider, for example, the $\mathcal{VSK}$ formula schema $\mathcal{V}\varphi \Rightarrow \mathcal{S}\varphi$, which says that if the information φ is accessible, then the agent perceives φ. Intuitively, this axiom characterises agents equipped with "perfect" sensors, i.e., sensors that obtain all the information from the environment that is available. In the following, we present results that correspond exactly to this and other intuitions. In addition, we give an axiomatisation of $\mathcal{VSK}$ logic, which we prove to be complete with respect to our model of agents and environments.

2 A Semantic Framework

In this section, we present a semantic model of agents and the environments they occupy. This model plays the role in $\mathcal{VSK}$ logic that *interpreted systems* play in epistemic logic [2, pp103–107] — when we later prove completeness of a $\mathcal{VSK}$ axiomatisation, we prove it with respect to this semantic model. We begin by defining the components modelling the environment; we then define our model of agents; and finally, we combine these to give the notion of a $\mathcal{VSK}$ system. A visual representation of the framework is given in Figure 1.

Following [2], we use the term "environment" to denote all the components of a system external to the agent. Sometimes environments can be represented as just another agent of the system; more often they serve a special purpose, as they can be used to model communication architectures, etc. We model an environment as a 4-tuple containing a set of possible *instantaneous states*, a *visibility function*, which characterises the information content of any given environment state, a *state transformer* function, which characterises the effects that an agent's actions have on the environment, and,

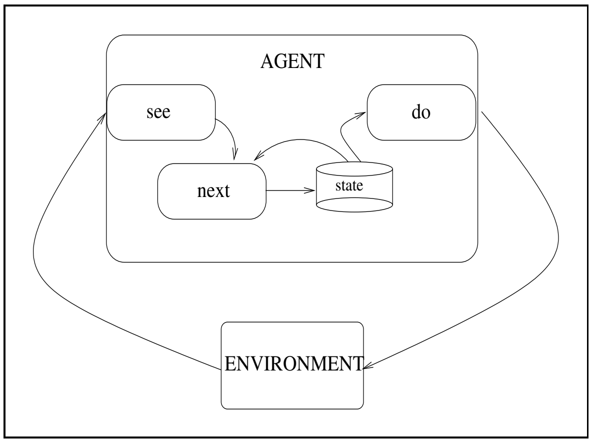

Figure 1: An overview of the framework.

finally, an *initial state*.

Definition 1 (Environments) *An* environment *is a tuple* $Env = \langle E, vis, \tau_e, e_0 \rangle$*, where:*

- $E = \{e_1, e_2, \ldots\}$ *is a* set of instantaneous local states *for the environment.*
- $vis : E \to 2^E$ *is the* visibility function of the $\mathcal{VSK}$ system*. It is assumed that the function vis partitions E into mutually disjoint sets and that* $e \in vis(e)$*, for any* $e \in E$*. Elements of the codomain of the function vis are called* visibility sets*. We say that vis is* transparent *if for any* $e \in E$ *we have that* $vis(e) = \{e\}$*.*
- $\tau_e : E \times Act \to E$ *is a total* state transformer function *for the environment (cf. [2, p154]), where Act is the set of* actions *for the agent (see Definition 2). The function* τ_e *is assumed to be an injection.*
- $e_0 \in E$ *is the* initial state *of Env.*

Modelling an environment in terms of a set of states and a state transformer is quite conventional (see, e.g., [2]). The only point worthy of note is that we implicitly assume environments evolve *deterministically*: there is no uncertainty about the result of performing an action in some state. The use of the visibility function, however, requires some explanation. The visibility function defines what is in principle knowable about a $\mathcal{VSK}$ system; the idea is similar to the notion of "partial observability" in POMDPs [7]. Intuitively, not all the information in an environment state is in general accessible to an agent. So, in a global state $g = (e, l)$, $vis(e) = \{e, e', e''\}$ represents the fact that the environment states e, e', e'' are indistinguishable to the agent from e. This is so regardless of the agent's efforts in performing the observation — it represents the maximum amount of information that is in principle available to the agent when observing the state e. The concept of transparency in Definition 4 captures "perfect" scenarios, in which all the information in a state is accessible to an agent. Note that visibility functions are *not* intended to capture the everyday notion of visibility as in "object x is visible to the agent".

We adopt a simple, and, we argue, general model of agents, which makes only a minimal commitment to an agent's internal architecture. One important assumption we do make is that agents have an internal state, although we make no assumptions with respect to the actual structure of this state. Agents are assumed to be composed of three functional components: some sensor apparatus, an action selection function, and a next-state function.

Definition 2 (Agents) *An* agent *is a tuple* $Ag = \langle L, Act, see, do, \tau_a, l_0 \rangle$*, where:*

- $L = \{l_1, l_2, \ldots\}$ *is a set of* instantaneous local states *for the agent.*
- $Act = \{\alpha, \alpha', \ldots\}$ *is a set of* actions*.*
- $see : vis(E) \to Perc$ *is the* perception function*, mapping visibility sets to* percepts*. Elements of the set Perc will be denoted as* $\rho, \rho', \ldots$ *and so on. If see is an injection into Perc then we say that see is* perfect*, otherwise we say it is* lossy*.*
- $do : L \to Act$ *is the* action selection function*, mapping local states to actions.*
- $\tau_a : L \times Perc \to L$ *is the* state transformer function *for the agent. We say that* τ_a *is* complete *if for any global states* $g = (e, \tau_a(l, \rho)), g' = (e', \tau_a(l', \rho'))$ *we have that* $\tau_a(l, \rho) = \tau_a(l', \rho')$ *implies* $\rho = \rho'$*, for every* $l, l' \in L'; e, e' \in E; \rho, \rho' \in Perc$*. We say that* τ_a *is* local *if for any global states* $g = (e, \tau_a(l, \rho))$*,* $g' = (e', \tau_a(l', \rho))$ *we have that* $\tau_a(l, \rho) = \tau_a(l', \rho)$ *for every* $l, l' \in L; e, e' \in E; \rho \in Perc$*. We say that an agent has* perfect recall *if the function* τ_a *is an injection.*
- $l_0 \in L$ *is the* initial state *for the agent.*

Perfect perception functions distinguish between all visibility sets; lossy perception functions are so called because they can map different visibility sets to the same percept, thereby losing information. We say that an agent has *perfect recall* of its history if it changes its local state at every tick of the clock (cf. [2, pp128–131]). Perfect recall is a very strong property to demand of agents, as it requires that they can distinguish every possible configuration of the system. As we will see below, perfect recall leads to an agent having perfect knowledge of the system, which is not likely to be possible in practice. Note that we have implicitly made the simplifying assumption that environment evolve synchronously with the agent.

We now require a working definition of the states of a $\mathcal{VSK}$ system, or *global states*.

Definition 3 (Global states for a $\mathcal{VSK}$ system) *A set of* global states $G = \{g, g', \ldots\}$ *for a* $\mathcal{VSK}$ *system is a subset of* $E \times L$*.*

We do not rule out G being equal to the Cartesian product of E and L; when this happens, the $\mathcal{VSK}$ system is said to be in a *hypercube configuration* and it enjoys some special properties (see [10, 9] for details). We can now define $\mathcal{VSK}$ *systems*.

Definition 4 ($\mathcal{VSK}$ systems) *A $\mathcal{VSK}$* system *is a pair $S = \langle Env, Ag \rangle$, where Env is an environment, and Ag is an agent. The class of $\mathcal{VSK}$ systems is denoted by $\mathcal{S}$.*

Although the logics we discuss in this paper may be used to refer to *static* properties of knowledge, visibility, and perception, the semantic model naturally allows us to account for the temporal evolution of a $\mathcal{VSK}$ system. The behaviour of an agent situated in an environment can be summarised as follows. The agent starts in state l_0, the environment starts in state e_0. At this point the agent "synchronises" with the environment by performing an initial observation $see(vis(e_0))$, through the visibility function *vis*, and generates a percept $see(vis(e_0))$. The internal state of the agent is then updated, and becomes $\tau_a(l_0, see(vis(e_0)))$. The synchronisation phase is now over and the system starts its run from the initial state $g_0 = (e_0, \tau_a(l_0, see(vis(e_0)))$. An action $\alpha_0 = do(\tau_a(l_0, see(vis(e_0, l_0))))$ is selected and performed by the agent on the environment, whose state is updated into $e_1 = \tau_e(e_0, \alpha_0)$. The agent enters another cycle, and so on. A *run* of a system is thus a (possibly infinite) sequence of global states defined as follows.

Definition 5 (Runs) *A sequence $(g_0, g_1, g_2, \ldots)$ over G represents a run of an agent $Ag = \langle L, Act, see, do, \tau_a, l_0 \rangle$ in an environment $Env = \langle E, vis, \tau_e, e_0 \rangle$ if*

- *$g_0 = (e_0, \tau_a(l_0, see(vis(e_0))))$, and*
- *for all u, if $g_u = (e_u, l_u)$, then $g_{u+1} = (e_{u+1}, l_{u+1})$ is defined by:*

$$\begin{array}{rcl} e_{u+1} & = & \tau_e(e_u, do(l_u)) \quad \textit{and} \\ l_{u+1} & = & \tau_a(l_u, see(vis(e_{u+1}))). \end{array}$$

Note that, since τ_e is an injection, two global states with the same environment component never occur in a run.

Definition 6 (Reachable states) *Given a $\mathcal{VSK}$ system $S = \langle Env, Ag \rangle$ we say G is the set of global states* generated *by S if $g \in G$ if and only if g occurs in the run of S.*

When $S = \langle Env, Ag \rangle$ is clear from the context we will refer to the set G of global states generated by $S = \langle Env, Ag \rangle$ simply as the set of global states of the $\mathcal{VSK}$ system $S = \langle Env, Ag \rangle$. Note that since both agents and environments are deterministic, a $\mathcal{VSK}$ system has only a single run; in this, we differ from [2].

3 $\mathcal{VSK}$ Logic

We now introduce a language $\mathcal{L}$, which will enable us to represent the information properties of $\mathcal{VSK}$ systems. In particular, it will allow us to represent first what is true of the $\mathcal{VSK}$ system, then what is *visible*, or *knowable* of the system, then what an agent *perceives* of the system, and finally, what it *knows* of the system.

Definition 7 (Syntax of $\mathcal{VSK}$ Logic) *Given a set P of propositional atoms, the language $\mathcal{L}$ of $\mathcal{VSK}$ logic is defined by the following BNF grammar:*

$\langle wff \rangle$ ::= **true** | *any element of P* | $\neg\langle wff \rangle$ | $\langle wff \rangle \wedge \langle wff \rangle$ | $\mathcal{V}\langle wff \rangle$ | $\mathcal{S}\langle wff \rangle$ | $\mathcal{K}\langle wff \rangle$.

The modal operator "$\mathcal{V}$" allows us to represent the information that is instantaneously visible or knowable about the state of the system. Thus, suppose the formula $\mathcal{V}\varphi$ is true in some state $g \in G$. The intended interpretation of this formula is that the property φ is *knowable* of the environment when it is in state g — not only is φ true of the environment, but any agent equipped with suitable sensor apparatus would be able to perceive the information φ. To put it another way, $\mathcal{V}\varphi$ means that an impartial external observer would say that in its current state, the environment carried the information φ. If $\neg\mathcal{V}\varphi$ were true in some state, then *no* agent, no matter how good its sensor apparatus was, would be able to perceive φ.

The fact that something is visible in a $\mathcal{VSK}$ system does not mean that an agent actually sees it. What an agent *does* see is determined by its sensors. The modal operator "$\mathcal{S}$" will be used to represent the information that an agent "sees". The idea is as follows. Suppose an agent's sensory apparatus (represented by the *see* function in our semantic model above) was a video camera, and so the percepts being received by the agent take the form of a video feed. Then $\mathcal{S}\varphi$ means that an impartial observer would say that the video feed currently being supplied by the video camera carried the information φ — in other words, φ is true all situations where the agent received the same video feed.

Finally, $\mathcal{VSK}$ logic allows us to represent an agent's *knowledge*. We represent knowledge by means of a third modal operator, "$\mathcal{K}$". In line with the tradition that started with Hintikka [4], we write $\mathcal{K}\varphi$ to represent the fact that the agent has knowledge of the formula represented by φ. Our model of knowledge is that popularised by Halpern and colleagues [2]: an agent is said to know φ when in local state l, if φ is guaranteed to be true whenever the agent is in state l. As with the $\mathcal{V}$ and $\mathcal{S}$ modalities, knowledge is an *external* notion — an agent is said to know φ if an impartial, omniscient observer would say that the agent's state carried the information φ.

We now proceed to interpret our formal language. While it is entirely possible to do so directly with respect to $\mathcal{VSK}$ systems, we will find it beneficial to use Kripke semantics [8] in order to prove completeness of an axiomatisation. In particular, we will use Kripke frames defined by three relations on their support set.

Definition 8 (Kripke frames and models) *A* frame *F is a tuple $F = \langle W, R_{\mathcal{V}}, R_{\mathcal{S}}, R_{\mathcal{K}}\rangle$, where W is a non-empty set (whose elements are called* worlds*), and $R_{\mathcal{V}}, R_{\mathcal{S}}, R_{\mathcal{K}} \subseteq W \times W$ are binary relations on W. If all relations are equivalence relations, the frame is an* equivalence frame *and we write $\sim_v, \sim_s, \sim_k$ for $R_{\mathcal{V}}, R_{\mathcal{S}}, R_{\mathcal{K}}$.*

We can define a mapping from the class of $\mathcal{VSK}$ systems to the class of Kripke frames and we can make use of these images to interpret our formal language.

Definition 9 (Generated Kripke structures) *Given a $\mathcal{VSK}$ system $S = \langle Env, Ag\rangle$, the* Kripke frame $F_S = \langle W, \sim_v, \sim_s, \sim_k\rangle$ generated by *S is defined as follows:*

- *$W = G$, where G is the set of global states reachable by the system S,*
- *$\sim_v$ is defined by: $(e,l) \sim_v (e',l')$ if $e' \in vis(e)$,*
- *$\sim_s$ is defined by: $(e,l) \sim_s (e',l')$ if $see(vis(e)) = see(vis(e'))$,*

- $\sim_k$ *is defined by:* $(e,l) \sim_k (e',l')$ *if* $l = l'$.

The class of frames generated by the class of $\mathcal{VSK}$ system $\mathcal{S}$ will be denoted by $\mathcal{F}_{\mathcal{S}}$; similarly F_S will denote the frame generated by the system S. As might be expected, the generated frames are equivalence frames.

Lemma 1 *Given any $\mathcal{VSK}$ system $S \in \mathcal{S}$, the frame F_S generated by S is an equivalence frame.*

With Definition 9 we have effectively built a bridge between $\mathcal{VSK}$ systems and Kripke frames. In what follows, we assume the standard definitions of satisfaction and validity for Kripke frames and Kripke models defined by three relations on the support set — we refer the reader to [6, 3] for a detailed exposition of the subject. Following [2] and [9], we define the concepts of truth and validity on Kripke models that are *generated* by $\mathcal{VSK}$ systems.

Definition 10 (Satisfaction on $\mathcal{VSK}$ systems) *Given an interpretation $\pi : W \to 2^P$, we say that a formula $\varphi \in \mathcal{L}$ is satisfied at a point $g \in G$ on a $\mathcal{VSK}$ system S if the model $M_S = \langle F_S, \pi \rangle$ built on the generated frame F_S by use of π is such that $M_S \models_g \varphi$. The propositional connectives are assumed to be interpreted as usual and the modal operators $\mathcal{V}, \mathcal{S}, \mathcal{K}$ are assumed to be interpreted in the standard way (see for example [6]) by means of the equivalence relations $\sim_\nu, \sim_s$, and $\sim_k$ respectively.*

We are especially interested in the properties of a $\mathcal{VSK}$ system as a whole. The notion of validity is appropriate for this analysis.

Definition 11 (Validity on $\mathcal{VSK}$ systems) *A formula $\varphi \in \mathcal{L}$ is valid on a class $\mathcal{S}$ of $\mathcal{VSK}$ systems if for any system $S \in \mathcal{S}$, we have that $F_S \models \varphi$.*

4 Axiomatising $\mathcal{VSK}$ Systems

In this section we study various $\mathcal{VSK}$ systems from an axiomatic perspective. This analysis will let us explore in more detail the properties of visibility, knowledge, and perception of $\mathcal{VSK}$ systems. We begin by presenting correspondence results; we then report completeness of an axiomatisation with respect to the most general class of $\mathcal{VSK}$ systems.

Let us first note that the class $\mathcal{F}_{\mathcal{S}}$ of frames generated by $\mathcal{VSK}$ systems is a proper subclass of equivalence frames. Indeed, the following holds.

Lemma 2 *For any frame $F \in \mathcal{F}_{\mathcal{S}}$, we have $\sim_\nu \subseteq \sim_s$.*

Lemma 3 *$F_S \models \mathcal{S}\varphi \Rightarrow \mathcal{V}\varphi$ if and only if $\sim_\nu \subseteq \sim_s$.*

In view of these lemmas, any $\mathcal{VSK}$ system validates the formula below.

Corollary 1 *Given any S we have $S \models \mathcal{S}p \Rightarrow \mathcal{V}p$.*

Corollary 1 is in line with our intuitions about visibility and perception: it says an agent cannot see something that it is not visible.

We now proceed to give basic correspondence results (see [1] for a detailed exposition of the subject) for axioms relating visibility, perception, knowledge with respect to the architectural classes of $\mathcal{VSK}$ systems described in Section 2. Note that our correspondence results are not simply given with respect to the Kripke frames but to architectural features of $\mathcal{VSK}$ systems.

Lemma 4

1. *$S \models p \Rightarrow \mathcal{V}p$ if and only if the system S is transparent.*
2. *$S \models p \Rightarrow \mathcal{S}p$ if and only if the system S is transparent and the perception function of the agent Ag in S is perfect.*
3. *$S \models p \Rightarrow \mathcal{K}p$ if and only if the agent has perfect recall.*

Lemma 4 makes precise the intuition given in the semantics of $\mathcal{VSK}$ systems about transparency, perfect perfection, and perfect recall. In particular, in order for the agent to be able to perceive everything that is true, it is not enough for it to have a perfect perception function: it also needs to inhabit a system with a transparent visibility function. Lemma 4 also clarifies the consequences of an assumption of perfect recall on $\mathcal{VSK}$ systems. Thus an agent with perfect recall will be able to distinguish between every configuration of the system. This is a rather strong property, it implies that an agent knows everything that is true.

We now investigate interaction axioms between visibility, perception, and knowledge. First, recall from Corollary 1 that on any generated frame the implication between perception and visibility is valid. Here we turn to the converse direction: if a fact is visible, then it is seen by the agent — in other words, the agent sees everything visible. Intuitively, this axiom characterises agents with "perfect" sensory apparatus, i.e., a *see* function that *never loses information*. Indeed, as the next lemma shows, this axiom corresponds formally to the perception function of the agent being perfect (as defined in Definition 2).

Lemma 5 *$S \models \mathcal{V}p \Rightarrow \mathcal{S}p$ if and only if the perception function see of the agent Ag in S in perfect.*

Given Corollary 1, we can strengthen the above as follows.

Corollary 2 *$S \models \mathcal{V}p \Leftrightarrow \mathcal{S}p$ if and only if the perception function see of the agent Ag in S in perfect.*

Suppose we have an agent which assumes that if it cannot see φ, then φ must be false. Such an agent is employing a kind of strict closed world assumption. We formally analyse the contrapositive of it.

Lemma 6 *$S \models \neg\mathcal{S}\neg p \Rightarrow \mathcal{V}p$ if and only if system S is transparent and the visibility function is perfect.*

We now turn to the relationship between what an agent perceives and what it knows. Recall from Definition 2 that complete transformer functions characterise agents that never lose information when they update their internal state. The following holds.

Lemma 7 *$S \models \mathcal{S}p \Rightarrow \mathcal{K}p$ if and only if the state transformer function τ_a is complete.*

Suppose that an agent's internal state at any moment is determined *solely* by the percept it receives at that moment — the agent chooses its next state by ignoring its current local state, and only taking into account the percept that it is currently receiving. This is the *locality* property of the state transformer function τ_a as described in Definition 2. For such agents, knowledge would appear to be determined solely by the current state of the environment. Indeed, we report the following.

Lemma 8 *$S \models \mathcal{K}p \Rightarrow \mathcal{S}p$ if and only if the state transformer function τ_a of system S is local.*

So far we have identified certain classes of $\mathcal{VSK}$ systems. In particular we were able to report that some architectural features of particular $\mathcal{VSK}$ systems are reflected in the validity of some axioms expressing implications between visibility, perception, and knowledge. We now turn our attention to the issue of completeness.

Many different $\mathcal{VSK}$ systems are worth exploring. As discussed above, the environment can be transparent or not, the agent's perception function can be perfect or otherwise, the agent's next state function can be complete, local or neither of the two, and so on. While we reported correspondence results, these are in general not enough to provide completeness, and each semantic class needs its own appropriate analysis. In this article we focus on a basic $\mathcal{VSK}$ logic: we prove that this logic axiomatises the most general class of $\mathcal{VSK}$ systems.

Definition 12 *The logic L_{VSK} is the set of formulas generated by the following axiomatisation.*

Taut	$\vdash_{\mathrm{L}_{VSK}} p$, *where p is any propositional tautology*
$K_{\mathcal{K}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{K}(p \Rightarrow q) \Rightarrow (\mathcal{K}p \Rightarrow \mathcal{K}q)$
$T_{\mathcal{K}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{K}p \Rightarrow p$
$4_{\mathcal{K}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{K}p \Rightarrow \mathcal{K}\mathcal{K}p$
$5_{\mathcal{K}}$	$\vdash_{\mathrm{L}_{VSK}} \neg\mathcal{K}p \Rightarrow \mathcal{K}\neg\mathcal{K}\neg p$
$K_{\mathcal{V}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{V}(p \Rightarrow q) \Rightarrow (\mathcal{V}p \Rightarrow \mathcal{V}q)$
$T_{\mathcal{V}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{V}p \Rightarrow p$
$4_{\mathcal{V}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{V}p \Rightarrow \mathcal{V}\mathcal{V}p$
$5_{\mathcal{V}}$	$\vdash_{\mathrm{L}_{VSK}} \neg\mathcal{V}p \Rightarrow \mathcal{V}\neg\mathcal{V}\neg p$
$K_{\mathcal{S}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{S}(p \Rightarrow q) \Rightarrow (\mathcal{S}p \Rightarrow \mathcal{S}q)$
$T_{\mathcal{S}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{S}p \Rightarrow p$
$4_{\mathcal{S}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{S}p \Rightarrow \mathcal{S}\mathcal{S}p$
$5_{\mathcal{S}}$	$\vdash_{\mathrm{L}_{VSK}} \neg\mathcal{S}p \Rightarrow \mathcal{S}\neg\mathcal{S}\neg p$
$Int_{\mathcal{S}-\mathcal{V}}$	$\vdash_{\mathrm{L}_{VSK}} \mathcal{S}p \Rightarrow \mathcal{V}p$
US	*If* $\vdash_{\mathrm{L}_{VSK}} \varphi$, *then* $\vdash_{\mathrm{L}_{VSK}} \varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$
MP	*If* $\vdash_{\mathrm{L}_{VSK}} \varphi$ *and* $\vdash_{\mathrm{L}_{VSK}} \varphi \Rightarrow \psi$, *then* $\vdash_{\mathrm{L}_{VSK}} \psi$

$Nec_{\mathcal{K}}$ *If* $\vdash_{\mathrm{L}_{VSK}} \varphi$, *then* $\vdash_{\mathrm{L}_{VSK}} \mathcal{K}\varphi$
$Nec_{\mathcal{V}}$ *If* $\vdash_{\mathrm{L}_{VSK}} \varphi$, *then* $\vdash_{\mathrm{L}_{VSK}} \mathcal{V}\varphi$
$Nec_{\mathcal{S}}$ *If* $\vdash_{\mathrm{L}_{VSK}} \varphi$, *then* $\vdash_{\mathrm{L}_{VSK}} \mathcal{S}\varphi$

It is immediately apparent that each of the $\mathcal{VSK}$ modalities enjoy the properties of an S5 modal logic: they each validate analogues of the modal logic axioms KT45 [6, 3]. The appropriateness of S5 as a logic of (idealised) knowledge has been discussed at length in the literature, and is now widely accepted [2, pp30–36]; for this reason, we will not motivate the S5 logic of knowledge. However, the appropriateness of S5 for the $\mathcal{V}$ and $\mathcal{S}$ modalities requires some justification.

Consider the $\mathcal{V}$ modality first. Recall the intended interpretation of a formula $\mathcal{V}\varphi$: that $\mathcal{V}\varphi$ is true in some state if an impartial observer would say that this state carried the information φ. Taking the axioms KT45 in turn, $K_{\mathcal{V}}$ seems unproblematic: if the information $p \Rightarrow q$ and p is carried by a state, then q must also be carried by that state. Axiom $T_{\mathcal{V}}$ simply says that if information p is carried by a state, then p must be true. This is a desirable property, since it would seem unreasonable to say that a state really carried some information if that information were false. Axiom $4_{\mathcal{V}}$ says that if we can conclude that a state carries information p, then we also have some additional (although arguably not terribly helpful) information: that it carries the information that it carries the information p. Since we have axiom $T_{\mathcal{V}}$, it follows that $\mathcal{V}p \Leftrightarrow \mathcal{V}\mathcal{V}p$ will be an axiom: we can remove repeated occurrences of the $\mathcal{V}$ modality without affecting the truth of a formula. Finally, axiom $5_{\mathcal{V}}$ says that if we can conclude that a state does not carry the information p, then we can conclude that the state carries the information that it does not carry the information p. Axioms $4_{\mathcal{V}}$ and $5_{\mathcal{V}}$ thus extend our information about a state from understanding the limits to the information carried by that state.

Turning to the $\mathcal{S}$ modality, we should first emphasise that $\mathcal{S}$ is *not* intended to form a logic of perception in the sense of, for example, Hintikka's [5, pp151–183]. Rather, $\mathcal{S}$ captures an *objective* notion of perception, (what an omniscient impartial observer would say you are seeing), rather than a *subjective* view of perception (what you believe you are seeing). Thus $\mathcal{S}\varphi$ means that if the agent is receiving some percept ρ, then whenever it receives percept ρ, formula φ is guaranteed to be true. In this sense, the percept the agent receives is carrying the information φ. We argue that under this interpretation, the S5 axioms capture reasonable properties of the $\mathcal{S}$ modality. The most controversial of these axioms for $\mathcal{S}$ is $T_{\mathcal{S}}$, and it is therefore worth examining this axiom in more detail. It says that if an agent "sees" p, then p must be true. If we were attempting to capture the everyday sense of *human* perception, then this axiom would not be acceptable — there are many obvious reasons why, if you perceive p, you could be wrong. However, under our interpretation, we say that $\mathcal{S}p$ means that in every state where you receive the same percept that you are currently receiving, p is true — in particular, p must be true in the current state. We can argue similarly for axioms $4_{\mathcal{S}}$ and $5_{\mathcal{S}}$.

We can prove that Definition 12 represents a sound and complete axiomatisation for the most general class of $\mathcal{VSK}$ systems.

Theorem 1 *The logic* L_{VSK} *is sound and complete with respect to the class of* $\mathcal{VSK}$ *systems* $\mathcal{S}$.

Proof: It is straightforward to show that L_{VSK} is sound with respect to the class of systems $\mathcal{S}$. In order to prove completeness, it suffices to show that $L_{VSK} \nvdash \varphi$ implies $\mathcal{S} \not\models \varphi$, for any $\varphi \in \mathcal{L}$. By carrying out a routine proof via the canonical model method (cf., e.g., [11]) one can show that the logic L_{VSK} is complete with respect to the class $\mathcal{G}$ of equivalence frames $F = \langle U, \sim_\nu, \sim_s, \sim_k \rangle$, where $\sim_\nu \subseteq \sim_s$. But, as we show below, given any frame $G = \langle W, \sim_\nu, \sim_s, \sim_k \rangle \in \mathcal{G}$, one can define a system $S \in \mathcal{S}$ such that its generated frame F_S is the domain of a p-morphism onto G.

Indeed, given the frame G above which we suppose defined on a countable set of worlds, consider the system $S = \langle Env, Ag \rangle$ defined as follows.

- For the environment component Env (see Definition 1):
  - $E = W$;
  - $vis : E \to 2^E$ defined by: $vis(w) = [w]_{\sim_\nu}$.
  - $\tau_e : E \times Act \to E$ defined as follows. Given a point w_i and an action α_j such that $j = min\{k \mid w_k \in [w_i]_{\sim_\nu}\}$, the target environment is defined by $\tau_e(w_i, \alpha_j) = w_{i+1}$. Note that τ_e is an injection as required.
  - $e_0 = w_1$.
- For the agent component Ag (see Definition 2):
  - $L = W/\sim_k$.
  - $Act = \{\alpha_1, \ldots, \alpha_n, \ldots\}$.
  - $Perc = W/\sim_s$. The function $see : vis(E) \to Perc$ is defined by $see([w]_{\sim_\nu}) = [w]_{\sim_s}$.
  - The function $do : L \to Act$ is defined by $do([w_i]_{\sim_k}) = \alpha_j, j = min\{k \mid w_k \in [w_i]_{\sim_k}\}$. Minimum index guarantees that do is well-defined.
  - $\tau_a : L \times Perc \to L$ is defined by $\tau_a([w_i]_{\sim_k}, [w_i]_{\sim_s}) = [w_{i+1}]_{\sim_k}$.
  - $l_0 = [w_1]_{\sim_k}$.

Given the above, the set of global states for system S are defined as $G = \{(w, [w]_{\sim_k}) \mid w \in W\}$.

By induction it is possible to show that every global state of G is eventually reached in the run r generated from $(w_1, [w_1]_{\sim_k})$. So the set G is effectively the set of reachable states (see Definition 6).

We have completed the description of system S. Consider now the frame $F_S = \langle W', \sim'_\nu, \sim'_s, \sim'_k \rangle$ generated by S according to Definition 9. Clearly, we have that F_S is an equivalence frame such that $\sim_\nu \subseteq \sim_s$. We define the naturally-induced p-morphism $p : F_S \to F$:

$$p((w, [w]_{\sim_k})) = w.$$

The mapping p is clearly well-defined and surjective. We prove that it follows the three properties of p-morphisms (see [3]).

- The function p is surjective. For any $w \in W$, we have the existence of a $(w, [w]_{\sim_k})$ such that $p((w, [w]_{\sim_k})) = w$.
- Suppose $g \sim_k' g'; g = (w, [w]_{\sim_k}), g' = (w', [w']_{\sim_k})$. But then it must be $[w]_{\sim_k} = [w']_{\sim_k}$ and therefore we have $w \sim_k w'$.

  Suppose $g \sim_v' g'; g = (w, [w]_{\sim_k}), g' = (w', [w']_{\sim_k})$. So we have that $w' \in [w]_{\sim_v}$, hence $w \sim_v w'$.

  Suppose $g \sim_s' g'; g = (w, [w]_{\sim_k}), g' = (w', [w']_{\sim_k})$. So we have that $see(vis((w, [w]_{\sim_k}))) = see(vis((w', [w']_{\sim_k})))$. So we have that $[w]_{\sim_s} = [w']_{\sim_s}$; but then $w \sim_s w'$.
- Consider any $(w, [w]_{\sim_k}) \in W'$ and a $v \in W$ such that $p((w, [w]_{\sim_k})) \sim_k v$. So $w \sim_k v$. But by construction we have that $(v, [v]_{\sim_k}) \in W'$ and, since $[w]_{\sim_k} = [v]_{\sim_k}$, we have $(w, [w]_{\sim_k}) \sim_k (v, [v]_{\sim_k})$. We also clearly have $p((v, [v]_{\sim_k})) = v$.

  Consider any $(w, [w]_{\sim_k}) \in W'$ and a $v \in W$ such that $p((w, [w]_{\sim_k})) \sim_v v$. So we have $w \sim_v v$, i.e. $[w]_{\sim_v} = [v]_{\sim_v}$. So $vis(w) = vis(v)$ and therefore $(w, [w]_{\sim_k}) \sim_v (v, [v]_{\sim_k})$. We also clearly have $p((v, [v]_{\sim_k})) = v$.

  Consider any $(w, [w]_{\sim_k}) \in W'$ and a $v \in W$ such that $p((w, [w]_{\sim_k})) \sim_s v$. So $w \sim_s v$ and so $[w]_{\sim_s} = [v]_{\sim_s}$. Since we have $p((v, [v]_{\sim_k})) = v$, it only remains to prove that $(w, [w]_{\sim_k}) \sim_v (v, [v]_{\sim_k})$, i.e. that $see(vis(w)) = see(vis(v))$. But we have $[w]_{\sim_s} = [v]_{\sim_s}$, and so the condition is verified.

Suppose then $\mathrm{L}_{VSK} \nvdash \varphi$, then by the completeness result above we have $G \not\models \varphi$ for some $G \in \mathcal{G}$; but then by constructing S as above, we can prove that, because of considerations on the transfer of validity to p-morphic images (e.g., see [3] page 11 for details on the mono-modal case), $F_S \not\models \varphi$. So $S \not\models \varphi$, hence $\mathcal{S} \not\models \varphi$, where $\mathcal{S}$ is the class of $\mathcal{VSK}$ systems. This is what we needed to show.

□

5 Conclusions

In this paper, we have introduced $\mathcal{VSK}$ logic as a formalism for representing and reasoning about the information properties of agents and their environments. Using $\mathcal{VSK}$ logic, we are able to represent what is objectively true of some environment, what is accessible or visible of the environment, what an agent sees of the environment, and finally, what an agent knows. The semantics of $\mathcal{VSK}$ logic were presented with respect to a simple and general model of agents and their environments. We were able to prove correspondence results for a number of possible axioms of $\mathcal{VSK}$ logic with respect to this model of agents and environments, thus demonstrating that certain axioms captured quite intuitive architectural properties of agent/environment systems. Finally, we gave an axiomatisation of $\mathcal{VSK}$ logic, and proved completeness of this logic with respect to the formal model of agents and environments. It is worth stressing that completeness was shown with the *grounded* semantics of Section 2 and that Kripke models are only used as a vehicle to achieve the result.

There are many avenues for future work: temporal extensions and multi-agent extensions are two of the most important. Completeness results for all basic $\mathcal{VSK}$ systems are another area of work. Finally, decidability and complexity results are desirable, perhaps by using the results of [2, pp62–76].

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